Initial and Boundary Conditions (continued from last time)

We now consider several other types of boundary conditions that can be associated with the wave equation. If the ends of the string move as $t$ increases according to some law, then we may have the inhomogeneous boundary conditions

$$u(0, t) = h_1(t), \quad u(L, t) = h_2(t) \quad \text{for } t > 0.$$ 

We may also consider a situation in which the ends are free to move transversally without any resistance. In this case, the (homogeneous) boundary conditions become

$$u_x(0, t) = u_x(L, t) = 0 \quad \text{for } t > 0.$$ 

If an external force is applied at the boundary, then these conditions may also become inhomogeneous:

$$u_x(0, t) = h_1(t), \quad u_x(L, t) = h_2(t) \quad \text{for } t > 0.$$ 

Finally, if the ends of the string are fixed to an elastic attachment that pulls the string back toward the equilibrium position, we may have boundary conditions of the form

$$\alpha_1 u(0, t) + \beta_1 u_x(0, t) = \alpha_2 u(L, t) + \beta_2 u_x(L, t) = 0 \quad \text{for } t > 0.$$ 

The Three Most Important Types of Boundary Conditions

We give special names to the types of boundary conditions we saw above in the wave equation example. They are as follows:

I. Dirichlet boundary conditions: $u$ is specified at the boundary

II. Neumann boundary conditions: $\frac{\partial u}{\partial n}$ is specified at the boundary

III. Robin boundary conditions: $\alpha u + \beta \frac{\partial u}{\partial n}$ is specified at the boundary

Note that within a given problem, the boundary conditions may be of different types on the various edges of the boundary.

We now present some examples of other types of problems associated with PDEs.

Ex: Consider the wave equation $u_{tt} = c^2 u_{xx}$ defined on the interval $-\infty < x < \infty$, $t > 0$. This may happen if we wish to ignore the effects on the boundary and are only concerned with the behavior of the model far away from the boundary. Then we do not need to specify any boundary conditions, although we still have the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

Such a problem is called an initial value problem (IVP) or a Cauchy problem.

Ex: We may also only be interested in the effects at one end of the boundary and consider the wave equation $u_{tt} = c^2 u_{xx}$ defined on a semi-infinite domain $x > 0$ with $t > 0$. In this case, we still need the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

as well as boundary conditions at the left edge $x = 0$. So we may set $u(0, t)$, $u_x(0, t)$, or $\alpha u(0, t) + \beta u_x(0, t)$ equal to 0 or some $h(t)$ depending on if we are using type I, II, or III boundary conditions.
If we consider a steady-state problem that is independent of time, then the problem is completely specified by the PDE and the boundary conditions. For example, we could have Poisson’s equation $\Delta u = F$ together with the boundary conditions

I. $u \big|_{\partial D} = h$

II. $\frac{\partial u}{\partial n} \big|_{\partial D} = h$

or

III. $\alpha u + \beta \frac{\partial u}{\partial n} \big|_{\partial D} = h$.

As a final example, we consider the heat equation $u_t = ku_{xx}$ for $0 < x < L$, $t > 0$. Then we only need to prescribe one initial condition $u(x, 0) = f(x)$. The boundary conditions may take the forms

I. $u(0, t) = u(L, t) = 0$ (the ends of the bar are held at 0 temperature)

II. $u_x(0, t) = u_x(L, t) = 0$ (the boundary is perfectly insulated and no heat may cross the boundary)

or

III. $\alpha_1 u(0, t) + \beta_1 u_x(0, t) = \alpha_2 u(L, t) + \beta_2 u_x(L, t) = 0$ (the ends of the bar lose heat at a rate proportional to the current temperature).

We could also have the inhomogeneous forms of any of the above conditions, or we could also consider the Cauchy problem for the heat equation and specify no boundary conditions.

Well-Posed Initial Boundary Value Problems

When solving a PDE together with a set of initial and/or boundary conditions, we wish the problem to be well-posed, i.e. satisfy the following conditions:

(i) Existence: There exists at least one solution $u(x, t)$ satisfying the PDE and the associated conditions.

(ii) Uniqueness: There is at most one solution.

(iii) Stability: If the data of the problem (initial/boundary conditions) are changed a little, the corresponding solution only changes a little as well.

This third condition is often required in models of physical problems. Since a data measurement is never made with infinite precision, the behavior of the solution should not be significantly affected by a slight perturbation of the given data.

Some Analytical Considerations

Consider the wave equation $u_{tt} = u_{xx}$ for $0 < x < 1$, $t > 0$. We observe that one solution of this equation is given by

$$u(x, t) = \sin(\pi x) \cos(\pi t).$$

Indeed, we may quickly verify $u_{tt} = -\pi^2 \sin(\pi x) \cos(\pi t) = u_{xx}$. What initial/boundary conditions does this solution satisfy?

For the initial conditions, we have that $u(x, 0) = \sin(\pi x)$ and $u_t(x, 0) = 0$. For the boundary conditions, we see that the solution satisfies $u(0, t) = u(1, t) = 0$. If we impose all of these conditions on the wave equation, then $u(x, t) = \sin(\pi x) \cos(\pi t)$ is the unique solution of the IBVP.

Let’s take a look at the solution at a few different time values to observe the vibrations:
Figure 1: Plots of \( u(x, t) = \sin(\pi x) \cos(\pi t) \) at various \( t \) values

Note: We also observe that \( u(x, t) = \sin(n\pi x) \cos(n\pi t) \) is a solution to the wave equation, as is \( u(x, t) = \sin(\pi x) \cos(\pi t) - \frac{1}{2} \sin(2\pi x) \cos(2\pi t) + \frac{1}{3} \sin(3\pi x) \cos(3\pi t) \). In the next section, we will state this observation in general for a linear, homogeneous PDE, but first we provide some additional graphs of these other more complicated solutions.

Figure 2: Plots of \( u(x, t) = \sin(n\pi x) \cos(n\pi t) \) for various \( n, t \) values
Figure 3: Plots of $u(x, t) = \sin(\pi x) \cos(\pi t) - \frac{1}{2} \sin(2\pi x) \cos(2\pi t) + \frac{1}{3} \sin(3\pi x) \cos(3\pi t)$ at various $t$ values