MA 401 Test 3 Answer Key

1. a) We have \( g_1(t) = 30 \), \( g_2(t) = 30 \), \( f(x) = 30 \sin(x) \).

Since \( 0 \leq \sin(x) \leq 1 \) for \( 0 \leq x \leq \pi \), take \( m = 0 \), \( M = 30 \).

\[
0 \leq g_1(t) g_2(t) f(x) \leq 30.
\]

By the maximum principle, it then follows that the solution \( u(x,t) \) is bounded by these same constants:

\[
0 \leq u(x,t) \leq 30.
\]

(b) \( u_5(x) = \frac{I_2-I_1}{L} x + I_1 = 30 \).

\[
v(x,t) = \sum_{n=1}^{\infty} b_n e^{-nt} \sin(nx),
\]

\[
b_n = \frac{2}{\pi} \int_0^{\pi} [30 \sin(x) - 30] \sin(nx) \, dx
\]

\[
= \frac{60}{\pi} \left[ \int_0^{\pi} \sin(x) \sin(nx) \, dx - \int_0^{\pi} \sin(x) \, dx \right]
\]

\[
= \frac{60}{\pi} \left[ \left\{ = 0 \text{ if } n \neq 1 \right\} \right. + \left. \frac{1}{n} \cos(nx) \bigg|_0^{\pi} \right]
\]

\[
b_n = \frac{60}{\pi} \left[ \frac{(-1)^n - 1}{n} \right] \text{ if } n \neq 1.
\]

\[
b_1 = \frac{60}{\pi} \left[ \frac{\pi}{2} - 2 \right] = 30 - \frac{120}{\pi}
\]

\[
u(x,t) = u_5(x) + v(x,t) = 30 + (30 - \frac{120}{\pi}) e^{-t} \sin(x) + \sum_{n=2}^{\infty} \frac{60(-1)^{n-1}}{n\pi} e^{-nt} \sin(nx)
\]

As \( t \to \infty \), \( v(x,t) \to 0 \), so \( u(x,t) \to 30 \).
2. a) Since \( f \) is odd, we immediately have \( A(w) = 0 \).

\[
B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt = \frac{2}{\pi} \int_{0}^{\infty} \sin(\omega t) dt
\]

\[
= -\left. \frac{2}{\pi \omega} \cos(\omega t) \right|_{0}^{\infty}
\]

\[
= \frac{2}{\pi \omega} (1 - \cos(\omega))
\]

The Fourier integral representation is then

\[
\frac{2}{\pi} \int_{0}^{\infty} \frac{1 - \cos(\omega)}{\omega} \sin(\omega x) d\omega
\]

converging to

\[
\left\{ \begin{array}{ll}
-1, & -1 < x < 0 \\
1, & 0 < x < 1 \\
-\frac{1}{2}, & x = -1 \\
\frac{1}{2}, & x = 1 \\
0, & \text{otherwise}
\end{array} \right.
\]

(10 pts)

b) Taking \( x = 1 \) in part a, we have \(-\) (5 pts for correct \( x \)-value)

\[
\frac{2}{\pi} \int_{0}^{\infty} \frac{1 - \cos(\omega)}{\omega} \sin(\omega) d\omega = \frac{1}{2}
\]

\[
\int_{0}^{\infty} \frac{1 - \cos(\omega)}{\omega} \sin(\omega) d\omega = \frac{\pi}{4}
\]
3. (a) Our solution is

\[ u(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin(nx) \sin(my), \]

where

\[ E_{mn} = -\frac{1}{\pi^2 (n^2 + m^2)} \int_0^1 \int_0^1 \sin(nx) \sin(my) \sin(nx) \sin(my) \, dy \, dx \]

\[ = -\frac{1}{\pi^2 (n^2 + m^2)} \left[ \int_0^1 \sin(my) \sin(my) \, dy \right] \left[ \int_0^1 \sin(nx) \, dx \right] \]

\[ = \begin{cases} 0 & \text{if } n \neq 1 \\ \frac{1}{2} & \text{if } n = 1 \end{cases} = -\frac{1}{\pi^2 m \cos(m \pi)} \]

\[ E_{m1} = -\frac{2}{\pi^3 (m^2 + 1)} \]

\[ u(x,y) = \sum_{m=1}^{\infty} \frac{2}{\pi^3 (m^2 + 1)} \sin(mx) \sin(y) \]

(b) Split the problem into 5 simpler ones:

\[ f_1(x) \]

\[ g_1(y) \]

\[ \Delta u = \sin(mx) \]

\[ = 0 \]

\[ \Delta u = 0 \]

\[ \Delta u = 0 \]

\[ \Delta u = 0 \]

\[ g_2(y) \]

\[ \Delta u = 0 \]

\[ \Delta u = 0 \]

\[ g_2(y) \]

\[ \Delta u = 0 \]

\[ \Delta u = 0 \]

\[ g_2(y) \]

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\[ g_2(y) \]

\[ \Delta u = 0 \]

\[ \Delta u = 0 \]

\[ g_2(y) \] Solve each simpler problem (the Poisson problem from part a and 4 Laplace problems) to obtain solutions \( u_1(x,y), \ldots, u_5(x,y) \).

By superposition, the sum of these 5 solutions will solve the original problem.

(10 pts - explanation of superposition)

(5 pts - correct statement of simpler problems)